The Structuralist Roots of Formalism: Hilbert's Early Views

Abstract

This paper argues that Hilbert's early views have not been understood and that a correct understanding of them reveals that Hilbert held a structuralist position, before he was a formalist.

At the turn of the 19th century Hilbert corresponded with Frege and he published *Grundlagen der Geometrie* which included the first example of our now entrenched model-theoretic reasoning. Despite the fact that Hilbert would not invent his proof theory until 1921, it has been assumed that his *conception* of meta-mathematical properties such as consistency and independence was already proof-theoretic. Indeed, every interpretation of Hilbert's early views have agreed on this point, Shapiro tells us, "For Hilbert, coherence is consistency, and by this he surely meant deductive, proof-theoretic consistency" (2005:69). This assumption is also made explicit in Shapiro (1996:95) and in the seminal expositions of the Frege-Hilbert controversy given by Resnick (1974:134) and Blanchette (1996:320).

I argue that this assumption must be rejected and that the only correct interpretation of Hilbert's early understanding of consistency is actually given by Frege. Frege correctly identifies that Hilbert does not take the relata of his consistency relation to be sentences – in line with a proof-theoretic approach – or to be fully determinate thoughts – in line with Frege's own approach – but *schematic propositions*. These are the semantic counterparts of the re-interpretable sentences Hilbert employs as his axioms in his *Grundlagen*. Thus I establish that Hilbert's conception is certainly *syntactic*, in that it is concerned with the logical form, and not with the meaning of determinate sentences. But it is not syntactic in anything like the modern proof-theoretic sense, which consists in the formal manipulation of sentences under a specified deductive system.

Recovering Hilbert's early conception of consistency is important because other aspects of his writings at this time show that Hilbert conceived of these schematic propositions structurally. He tells Frege,

> It is surely obvious that every theory is only a scaffolding (schema) of concepts together with their necessary connections, and that the basic elements can be thought of in any way one likes. E.g., instead of points, think of a system of love, law, chimney sweeps... which satisfies all axioms. (1895:42)

Here Hilbert explains that *any* objects can qualify as geometric objects so long as they fit the *scaffolding* expressed by his axioms. The other properties of these objects are irrelevant for the purposes of their acting as a point (cf. 1895:39). Thus Hilbert's conception of what it is to be a mathematical object is to have all and only those properties in virtue of which the structure is satisfied. This is tantamount to a form of structuralism on which all there is to being a basic element is to occupy a node in the conceptual scaffolding of a theory.

The influential axiomatisation that Hilbert gives in foundations characterises *a euclidean structure*. This moves away from the traditional idea, still up held by Frege, of an axiomisation aiming to capture *the* geometry of space. In this sense it was Hilbert's conception of a theory which made the structuralist position in mathematics possible for the first time. On this new way of thinking structures had greater ontological importance than objects in the sense that the properties and existence of objects were grounded in the consistency of the structure rather than having the consistency of the structure grounded in the existence of the geometric objects, as Frege did.

Avoiding reading Hilbert's later views too far into his early work thus reveals that the father of formalism initially pioneered a structuralist understanding of his model-theoretic methodology.

References

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