Formal computation as deduction

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Summary: It became clear by the early 1930s that steps of formal computation are also steps of formal deduction as defined by recursion equations and other similar principles of arithmetic. How this happened will be detailed out, from Grassmann's *Lehrbuch der Arithmetik* in 1861 to Hilbert and Bernays' *Grundlagen* in 1934.

The first steps towards a formal theory of computation were taken by Hermann Grassmann in 1861. His idea was to apply recursion, known earlier from combinatorics, to the most elementary parts of arithmetic, namely the definition of the arithmetic operations. The way was opened for inductive proofs of the basic laws of arithmetic. At the same time, computation turned into a sequence of equations derived by the recursion equations and principles of substitution. Hermann Hankel presents the idea in 1867:

> *In this way one finds through a recurrent procedure, one that goes on purely mechanically without any intuition, unequivocally every sum of two numbers.*

This proclamation is followed by a mechanical computation of the formula $7 + 5 = 12$ – a "smoking gun" of sorts that betrays a Kantian influence – through the writing of 17 equations that begins with $7 + 5 = 7 + (4 + 1) = (7 + 4) + 1$ and ends with $7 + 5 = 11 + 1 = 12$. Hankel's account is repeated almost *verbatim* in Ernst Schröder's widely read *Lehrbuch der Arithmetik und Algebra* of 1873, a book referred to by von Helmholtz, Dedekind, and others.

The step to formal deduction was taken in two quarters. In a proof in arithmetic, one usually writes the equations one after the other in a vertical succession. Frege wanted to indicate the logical dependences between the equations and invented for this purpose his two-dimensional formula language, "built after the one of arithmetic" as he wrote. Peano in his turn formalized proofs in arithmetic in terms of instances of his famous axioms and the rule of modus ponens, writing that "we use in arithmetic proofs the book of Grassmann."

Grassmann's recursive arithmetic resurfaced decades later in the famous paper of Skolem, published in 1923. Right thereafter, Hilbert, Bernays, Ackermann, and Sudan in Göttingen began to work on the topic. They first identified primitive recursion, then found that its bounds can be surpassed by the introduction of an iteration functional, Ackermann's example of a recursive function of higher type, and went on to determine the ordinal of primitive recursion (Sudan) and to classify various forms of recursion (Ackermann, Sudan). None of this is seen in the first summary of the results of Hilbert's school, the *Grundzüge der theoretischen Logik* published under Hilbert and Ackermann's name in 1928 and dedicated to pure logic. There is instead a wonderful, clear insight in the chapter dedicated to recursion of the *Grundlagen der Mathematik* on how the recursive definition of basic arithmetic operations turns computation into deduction (p. 290):

> If c is a numeral, the computation of the value of $f(a, \ldots, t, x)$ is already completely formalized through the derivation of the equation
>
> $$f(a, \ldots, t, x) = c$$
>
> We can reconstruct completely the recursive procedure of computation of finitary number theory in our formalism, through the deductive application of the recursion equations.

The hundred-page chapter on recursion in the *Grundlagen* was shaped by the collaboration between Bernays and Rosa Politzer in 1933, barely making it for the final version published in 1934.